
CONIC SECTIONS – THE CIRCLE

Why this chapter? Because in previous chapters on the circle, we used patterns to determine the equation of a circle. This chapter will prove beyond a doubt that our circle equation is always correct.

We'll need the **Distance Formula** in the plane for this chapter:

If (x_1, y_1) and (x_2, y_2) are points in the plane, then the distance, ***d***, between them is given by the formula:

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

(which equals $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$)



Patterns don't always pan out. Consider:

3, 5, 7, ...

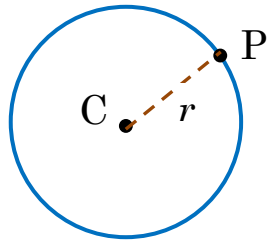
Is the next number **9**, because 3, 5, and 7 are consecutive *odd* numbers?

Or is the next number **11**, because they're consecutive *prime* numbers?

Who's to tell?

□ DEFINITION OF THE CIRCLE

Now for the official definition of the circle. Let C be any point in the plane, and let r be any positive real number. Then a ***circle*** is the set of points in the plane satisfying the following condition: *A point is on the circle if its distance to the point C is r .* The point C is called the **center** of the circle and the number r is called its **radius**.



If P is a point on the circle, then its distance to the center C must be r .

Equivalently:

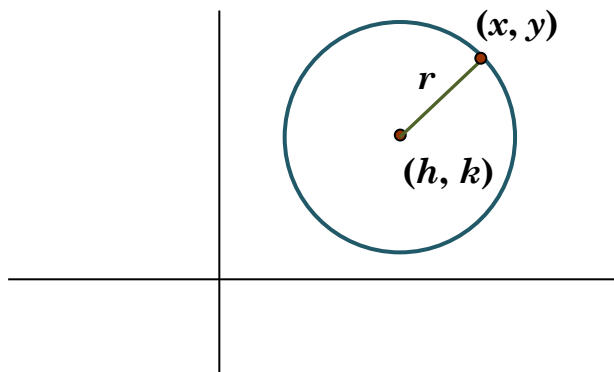
A **circle** is the set of all points in the plane which are *equidistant* from its center.

Homework

1. Why do we assume that r is a positive real number?
2. Let P be a point on a circle with center C , and assume that the distance between P and C is 18. Now let Q be another point on the circle. Find the distance between Q and C .

❑ THE EQUATION OF THE CIRCLE

We label the center of the circle (h, k) . And we use the generic point (x, y) to represent all the points on the circle.



r is the distance between the points (x, y) and (h, k) .

We note that the **distance** between the point (x, y) on the circle and the center (h, k) must be r , the radius of the circle. We can thus write, using the Distance Formula:

$$\sqrt{(x-h)^2 + (y-k)^2} = r$$

Squaring both sides of this equation produces our circle equation:

The equation of the circle with center (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

*The
Standard
Form of
a Circle*

EXAMPLE 1: Use the standard form of a circle to find the center and radius of the circle

$$(x-2)^2 + (y+7)^2 = 10$$

Solution: Transform the given equation into the following form, designed to match the standard form:

$$(x - \boxed{2})^2 + (y - (\boxed{-7}))^2 = (\boxed{\sqrt{10}})^2$$

Standard Form: $(x - \boxed{h})^2 + (y - \boxed{k})^2 = \boxed{r}^2$

We can deduce: $h = 2, \quad k = -7, \quad r = \sqrt{10}$

And we have our center and radius:

$$C(2, -7) \quad r = \sqrt{10}$$

EXAMPLE 2: Find the equation of the circle in standard form if the center of the circle is $(9, -4)$ and the radius is 5.

Solution: From the given center and radius, we get the values

$$h = 9, \quad k = -4, \quad r = 5$$

So, if we take the standard form of a circle:

$$(x - h)^2 + (y - k)^2 = r^2$$

and then place the three values into their appropriate positions, we get

$$(x - (9))^2 + (y - (-4))^2 = 5^2$$

Simplify, and we're done:

$$(x + 9)^2 + (y - 4)^2 = 25$$

Homework

3. Use the standard form of a circle to find the center and radius of each circle:
 - a. $x^2 + y^2 = 64$
 - b. $x^2 + y^2 = 24$
 - c. $x^2 + (y - 7)^2 = 4$
 - d. $(x - 3)^2 + y^2 = 5$
 - e. $(x + 1)^2 + (y + 8)^2 = 60$
 - f. $(x - 2)^2 + (y - 3)^2 = 99$
 - g. $(x + 5)^2 + (y - 3)^2 = 144$
 - h. $(x - 1)^2 + (y + 11)^2 = 48$
 - i. $(x + 3)^2 + (y - 1)^2 = 9$
 - j. $(x - 3)^2 + (y + 2)^2 = 7$

4. Find the equation of the circle with the given center and radius:

- | | | | |
|-------------|-----------------|--------------|-----------------|
| a. C(0, 0) | $r = 7$ | b. C(0, 0) | $r = \sqrt{10}$ |
| c. C(3, 0) | $r = 3$ | d. C(0, 4) | $r = 1$ |
| e. C(0, -2) | $r = \sqrt{3}$ | f. C(-12, 0) | $r = 12$ |
| g. C(2, 7) | $r = 10$ | h. C(-1, -3) | $r = 2\sqrt{3}$ |
| i. C(3, -4) | $r = 3\sqrt{5}$ | j. C(-2, 9) | $r = 5\sqrt{7}$ |
| k. C(1, 2) | $r = 0$ | l. C(-3, 5) | $r = -9$ |

Review Problems

5. Give the geometric definition of the circle.
6. Consider the equation $x^2 + y^2 = k$. Describe the graph of this equation if $k > 0$, if $k = 0$, and if $k < 0$.
7. Find the center and radius of the circle $x^2 + y^2 = 20$.
8. Find the center and radius of the circle $x^2 + y^2 - 10x + 2y + 3 = 0$.
9. In terms of exponents and coefficients, explain why each of the following is not a circle:

a. $2x + 3y = 1$	b. $y = x^2$	c. $y = x^3$
d. $x^2 + 3y^2 = 4$	e. $x^2 - y^2 = 9$	f. $y = \sqrt{x}$
10. True/False:
 - a. Every circle has at least one intercept.
 - b. The radius of the circle $x^2 + y^2 = 1$ is 1.

- c. The derivation of the circle formula is based upon the Distance Formula.
- d. $x^2 + y^2 + 4 = 0$ is a circle.
- e. $x^2 + y^2 = 1$ is called the unit circle.
- f. The center of the circle $(x - 2)^2 + (y - 5)^2 = 10$ is $(-2, -5)$.
- g. The radius of the circle $x^2 + y^2 + 8x - 6y + 9 = 0$ is 4.
- h. The radius of a circle whose diameter is $25\sqrt{2}$ is $5\sqrt{2}$.
- i. The graph of $10x^2 + 10y^2 = 37$ is a circle.
- j. The graph of $10x^2 - 10y^2 = 37$ is a circle.
- k. The graph of $10x^2 + 9y^2 = 37$ is a circle.
- l. The area of the circle $x^2 + y^2 = 25$ is 25π .

Solutions

- 1. r represents the radius (which is a distance), and thus makes sense only if $r > 0$ [a ‘circle’ with zero or negative radius wouldn’t be a circle].
- 2. It must be 18, since the distance from any point on the circle to the center must be the radius, which is 18.
- 3.

a. C(0, 0) $r = 8$	b. C(0, 0) $r = 2\sqrt{6}$
c. C(0, 7) $r = 2$	d. C(3, 0) $r = \sqrt{5}$
e. C(-1, -8) $r = 2\sqrt{15}$	f. C(2, 3) $r = 3\sqrt{11}$
g. C(-5, 3) $r = 12$	h. C(1, -11) $r = 4\sqrt{3}$
i. It’s a line	j. It’s a parabola
- 4.

a. $x^2 + y^2 = 49$	b. $x^2 + y^2 = 10$
c. $(x - 3)^2 + y^2 = 9$	d. $x^2 + (y - 4)^2 = 1$
e. $x^2 + (y + 2)^2 = 3$	f. $(x + 12)^2 + y^2 = 144$

- g. $(x - 2)^2 + (y - 7)^2 = 100$ h. $(x + 1)^2 + (y + 3)^2 = 12$
 i. $(x - 3)^2 + (y + 4)^2 = 45$ j. $(x + 2)^2 + (y - 9)^2 = 175$
 k. Just the point $(1, 2)$ l. Not a circle; no graph at all

5. A circle is the set of all points in the plane that are equidistant from a fixed point in the plane.

6. If $k > 0$, the graph is a circle with center at the origin and radius $\sqrt{k}$.
 If $k = 0$, the graph is just the single point $(0, 0)$; i.e., the origin.
 If $k < 0$, the graph is empty (there's no graph at all).

7. $C(0, 0); r = 2\sqrt{5}$

8. $(x - 5)^2 + (y + 4)^2 = 25$

9. a. $2x + 3y = 1$: At the very least, a circle has both variables squared, which the given equation certainly does not have. In fact, the given equation is a line.
- b. $y = x^2$: Again, at the very least, both variables need to be squared to be a circle. Indeed, the equation is a parabola.
- c. $y = x^3$: This is a cubic function.
- d. $x^2 + 3y^2 = 4$: This equation does have both variables squared, but the coefficients of the squared terms (the 1 and the 3) are not the same, so there's no way we could put the equation into the standard form of the circle: $(x - h)^2 + (y - k)^2 = r^2$. The chapter *Ellipses* will show you what the given equation actually looks like.
- e. $x^2 - y^2 = 9$: This equation resembles a circle's equation, but the minus sign between the squared terms kills the deal. Remember that the equation of a circle ultimately comes from the Distance Formula, which has a plus sign between the squared parts. We'll work with formulas like this in the chapter *Hyperbolas*.
- f. $y = \sqrt{x}$: Certainly the exponents aren't appropriate for a circle, since the y has an exponent of 1 and the x has an exponent of $1/2$.

In fact, the graph of this function is the top half of a parabola opening to the right.

11. a. F b. T c. T d. F e. T f. F
 g. T h. F i. T j. F k. F l. T

Only those who dare to fail greatly
can ever achieve greatly.

Robert F. Kennedy